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Intrinsic characterisation of orthogonal separation of one coordinate in the Hamilton–Jacobi equation

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Abstract. We extend an idea due to Woodhouse to give a coordinate-free characterisation of the orthogonal separation of one coordinate for the Hamilton–Jacobi equation on a pseudo-Riemannian manifold, in terms of an involutive family of Killing tensors. The coordinates can be computed from the Killing tensors.

1. Introduction

Let V^n be an n-dimensional pseudo-Riemannian manifold and let (y^i) be a local coordinate system for V^n. The Hamilton–Jacobi equation for V^n is

$$\mathscr{H}(\partial_i u) = \sum_{i,j=1}^{n} g^{ij}\partial_i u \partial_j u = E \tag{1.1}$$

where $\partial_i = \partial_{y^i}$, E is a constant, and the metric on V^n, expressed in the coordinates (y^i), is

$$\mathrm{d}s^2 = \sum_{i,j} g_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j. \tag{1.2}$$

Here, $\Sigma_k g_{ik} g^{kj} = \delta_i^j$. Equation (1.1) has the same form in all coordinate systems. All functions on V^n that occur in this paper are assumed to be locally analytic. Associated with V^n we have the $2n$-dimensional cotangent bundle $\tilde{V}^n$ with local coordinates (y^i, p_j). If $(y^{i'})$ is another coordinate system on V^n, $y^{i'} = f^i(y^l)$, it induces a new local coordinate system $(y^{i'}, p'_j)$ on $\tilde{V}^n$ where $p'_j = \Sigma_l\,(\partial y^l/\partial y^{j'})p_l$. Thus the Hamiltonian $\mathscr{H}$,

$$\mathscr{H}(\boldsymbol{p}) = \sum g^{ij}(y)p_i p_j, \tag{1.3}$$

is also independent of coordinates. Important to our discussion is the concept of a constant of the motion $\mathscr{L}(y, \boldsymbol{p})$, i.e. a function $\mathscr{L}$ on $\tilde{V}^n$ such that $\{\mathscr{L}, \mathscr{H}\} = 0$, where $\mathscr{H}$ is the Hamiltonian and

$$\{\mathscr{F}(y, \boldsymbol{p}), \mathscr{S}(y, \boldsymbol{p})\} = \sum_i (\partial_{p_i}\mathscr{F}\partial_{y^i}\mathscr{S} - \partial_{p_i}\mathscr{S}\partial_{y^i}\mathscr{F}) \tag{1.4}$$

is the Poisson bracket of functions $\mathscr{F}$ and $\mathscr{S}$ on $\tilde{V}^n$. If $\mathscr{L}$ is a polynomial in the p_i, then it is a Killing tensor. A Killing tensor linear in the p_i is a Killing vector.

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Woodhouse (1975) studied a simple construction that associates a constant of the motion for (1.1) with each coordinate system in which a single variable separates. He considered only the case where the separated coordinate x^1 was non-null, i.e. $g^{11} \neq 0$. For this case the associated constants of the motion are Killing vectors or second-order Killing tensors. (In Kalnins and Miller (1982) the authors present some preliminary results concerning the separation of a null coordinate.) He showed that non-null separation was of two types. One type corresponds to Killing vectors alone and is associated with *ignorable* separable coordinates, i.e. coordinates $x^1, \ldots, x^k$ such that $\partial_{x^l} g^{ij}(\boldsymbol{x}) = 0$ for $1 \leq l \leq k \leq n$. The second, and more interesting, type corresponds to second-order Killing tensors. Woodhouse showed that, for proper Riemannian and Lorentzian spaces at least, this second type was always equivalent to the existence of a non-null separable coordinate x^1 which was *orthogonal*, i.e. there exists a coordinate system (x^1, x^α) such that (1.1) has a complete integral

$$u(\boldsymbol{x}, \boldsymbol{c}) = u^{(1)}(x^1, \boldsymbol{c}) + u^*(x^\alpha, \boldsymbol{c}) \tag{1.5}$$

with $g^{1\alpha}(\boldsymbol{x}) = 0$, $2 \leq \alpha \leq n$.

In this paper we shall determine an intrinsic (coordinate-free) characterisation of orthogonal separation of one coordinate on any space V^n in terms of the Killing tensors associated with the coordinate by the Woodhouse construction. Such a characterisation of partial separation in terms of Killing tensors is of interest in celestial mechanics and general relativity theory (Woodhouse 1975, Benenti and Francaviglia 1980), and when extended to Laplace, wave, Helmholtz and Schrödinger equations it is useful in the construction of explicit solutions for the differential equations of mathematical physics (Kalnins *et al* 1979).

In § 2 we first review the relationship between complete integrals of (1.1), permitting separation of a single orthogonal coordinate, and the Woodhouse construction. We then derive the canonical set of separation equations satisfied by the complete integral. Our principal result is theorem 1, which gives a test for separation of the orthogonal coordinate x^1 in (x^i) in terms of a system of differential equations satisfied by the metric $g^{ij}(\boldsymbol{x})$. Section 3 is devoted to the proof of theorem 2, an intrinsic characterisation of orthogonal separation in terms of Killing tensors. For spaces in which the Killing tensors are just polynomials in the Killing vectors, such as spaces of constant curvature, theorem 2 reduces the study of single orthogonal variable separation to a strictly algebraic problem.

A similar treatment of *complete* orthogonal separation of (1.1) was given by the authors in Kalnins and Miller (1980).

2. The canonical separation equations

We say that a function $u = u(x^1, \ldots, x^n, c_1, \ldots, c_n)$ is a *complete integral* of the Hamilton–Jacobi equation

$$\mathscr{H}(\boldsymbol{x}, \boldsymbol{p}) \equiv \sum_{i,j=1}^{n} g^{ij}(\boldsymbol{x}) p_i p_j = E \tag{2.1}$$

provided u satisfies (2.1) with $p_i = \partial_{x^i} u$, that $c_1 = E$ and that the $n \times n$ matrix $(\partial p_i / \partial c_l)$ has rank n. (Here, the parameters $(c_1, \ldots, c_n)$ range over a connected open set in R^n.) As is well known (Woodhouse 1975), a complete integral is always associated with an involutive family of n constants of the motion.

Lemma 1. Let $u(\boldsymbol{x}, \boldsymbol{c})$ be a complete integral for the Hamilton–Jacobi equation. Then there exist n functions $\lambda_l(\boldsymbol{x}, \boldsymbol{p})$ on the cotangent bundle $\tilde{V}^n$ such that

(1) the $n \times n$ matrix $(\partial\lambda_l/\partial p_i)$ is non-singular;
(2) $\{\lambda_l, \lambda_m\} = 0$, $1 \leq l, m \leq n$;
(3) $\lambda_1 \equiv \mathcal{H}$;
(4) $\lambda_l(\boldsymbol{x}, \boldsymbol{p}) = c_l$, $1 \leq l \leq n$, where $p_i = \partial_{x^i} u(\boldsymbol{x}, \boldsymbol{c})$.

Conversely, if (λ_l) is a family of n functions on $\tilde{V}^n$ satisfying properties (1)–(3), then there exists a complete integral u for (2.1) satisfying property (4).

More generally, for $1 \leq k \leq n$ we have

Lemma 2. Let $\lambda_l(\boldsymbol{x}, \boldsymbol{p})$, $1 \leq l \leq k$, be a family of functions on $\tilde{V}^n$ such that

(1) the $l \times n$ matrix $(\partial\lambda_l/\partial p_i)$ has rank k;
(2) $\{\lambda_l, \lambda_m\} = 0$, $1 \leq l, m \leq k$;
(3) $\lambda_1 \equiv \mathcal{H}$.

Then there exists a complete integral $u = u(\boldsymbol{x}, \boldsymbol{c})$ such that

$$\lambda_l(x, p) = c_l, \qquad 1 \leq l \leq k, \qquad p_j = \partial u/\partial x^j. \tag{2.2}$$

Proof. Suppose the family $\{\lambda_l\}$ satisfies hypotheses (1)–(3). For a suitable choice of parameters c_l we can use condition (1) to show that the equations $\lambda_l(\boldsymbol{x}, \boldsymbol{p}) = c_l$, $1 \leq l \leq k$, can be solved for k of the p_i, which without loss of generality we can assume to be $p_1, p_2, \ldots, p_k$:

$$p_l = f_l(x, p_{k+1}, \ldots, p_n, \boldsymbol{c}), \qquad 1 \leq l \leq k. \tag{2.3}$$

Next we must show that there exists a function $u(\boldsymbol{x}, \boldsymbol{c})$ satisfying (2.3), where $p_i = \partial_{x^i} u$. For fixed x_0^l, $1 \leq l \leq k$ and a given function $u_0(x^{k+1}, \ldots, x^n; c_{k+1}, \ldots, c_n)$ we claim that there is a unique solution $u(\boldsymbol{x}, \boldsymbol{c})$ of equations (2.3) such that

$$u(x_0^1, \ldots, x_0^k, x^{k+1}, \ldots, x^n; \boldsymbol{c}) = u_0(x^{k+1}, \ldots, x^n; c_{k+1}, \ldots, c_n). \tag{2.4}$$

(We require that the $(n-k) \times (n-k)$ matrix $(\partial^2 u/\partial x^i \partial c_j)$ is non-singular, where $k \leq i, j \leq n$.)

The proof is a slight variant of the standard proof for the Cauchy–Kowalewski theorem (John 1978, Courant and Hilbert 1962). In particular the solution can be represented as a Taylor expansion

$$u(\boldsymbol{x}, \boldsymbol{c}) = \sum_{l_i \geq 0} \frac{U_{l_1,\ldots,l_k}}{l_1! \ldots l_k!}(x^{k+1}, \ldots, x^n)(x^1 - x_0^1)^{l_1} \ldots (x^k - x_0^k)^{l_k},$$

where the coefficients

$$U_{l_1,\ldots,l_k} = \partial_{x^1}^{l_1} \ldots \partial_{x^k}^{l_k} u(\boldsymbol{x}, \boldsymbol{c})|_{(x^j = x_0^j,\ 1 \leq j \leq k)}$$

are computed from (2.4) and successive differentiation of (2.3). The existence and uniqueness proofs are identical to the standard Cauchy–Kowalewski proofs except that it is necessary to show that whenever a coefficient $U_{l_1,\ldots,l_k}$ can be computed in two distinct ways from equations (2.3) the answers must agree. From the right-hand side of (2.2) we have

$$\frac{\partial p_m}{\partial x^l}-\frac{\partial p_l}{\partial x^m}=\sum_{a=1}^{k}\sum_{\alpha=k+1}^{n} D^l_a\frac{\partial\lambda_a}{\partial p_\alpha}\left(\frac{\partial p_\alpha}{\partial x^m}-\frac{\partial p_m}{\partial x^\alpha}\right)+\sum_{b=1}^{k}\sum_{\beta=k+1}^{n} D^m_b\frac{\partial\lambda_b}{\partial p_\beta}\left(\frac{\partial p_l}{\partial x^\beta}-\frac{\partial p_\beta}{\partial x^\alpha}\right)$$

$$+\sum_{a,b=1}^{k}\sum_{\alpha,\beta=k+1}^{n} D^l_a D^m_b\frac{\partial\lambda_a}{\partial p_\alpha}\frac{\partial\lambda_b}{\partial p_\beta}\left(\frac{\partial p_\alpha}{\partial x^\beta}-\frac{\partial p_\beta}{\partial x^\alpha}\right) \tag{2.5}$$

where $\Sigma_{a=1}^{k} D^l_a(\partial\lambda_a/\partial p_m)=\delta^l_m$, $1\leq l, m\leq k$. Evaluating this expression for $x^i=x^i_0$, $1\leq i\leq k$ we see that by construction $\partial p_\alpha/\partial x^\beta=\partial p_\beta/\partial x^\alpha$ and, since $\partial p_l/\partial x^\beta$ can be computed in only one way, $\partial p_l/\partial x^\beta=\partial p_\beta/\partial x^l$. Thus $\partial p_m/\partial x^l=\partial p_l/\partial x^m$ at $x^i=x^i_0$, $1<i<k$. Successive differentiation of (2.5) with respect to the variables x^i and use of (2.3) leads to the equality of all cross partials. Furthermore, $\det(\partial p_i/\partial c_j)\neq 0$ so u is a complete integral.

Suppose x^1 is an *orthogonal coordinate*, i.e. $\boldsymbol{x}=(x^1, x^2, \ldots, x^n)$ and $g^{1\alpha}(\boldsymbol{x})=0$ for $2\leq\alpha\leq n$, so that $g^{11}(\boldsymbol{x})\neq 0$. We say that the Hamilton–Jacobi equation is *separable* in the coordinate x^1 provided there exists a complete integral for (2.1) of the form

$$u(\boldsymbol{x},\boldsymbol{c})=u^{(1)}(x^1,\boldsymbol{c})+u^*(x^\alpha,\boldsymbol{c}). \tag{2.6}$$

Woodhouse (1975) showed that if x^1 is an orthogonal separable coordinate then

$$\mathscr{G}_\tau=\frac{-\mathscr{H}(x^1,x^\gamma,\boldsymbol{p})+\Sigma g^{\alpha\beta}(\tau,x^\gamma)p_\alpha p_\beta}{g^{11}(\tau,x^\gamma)} \tag{2.7}$$

is a function $\mathscr{G}_\tau=g_\tau(\lambda_1,\ldots,\lambda_n)$ of the involutive family λ_j where here the Hamiltonian takes the form

$$\mathscr{H}(x,p)\equiv g^{11}(\boldsymbol{x})p_1^2+\sum_{\alpha,\beta=2}^{n} g^{\alpha\beta}(\boldsymbol{x})p_\alpha p_\beta. \tag{2.8}$$

(Indeed $\mathscr{G}_\tau=-p_1^2(\tau,\lambda)$ for $x^1=\tau$.) Thus,

$$\{\mathscr{H},\mathscr{G}_\tau\}=\{\mathscr{G}_{\tau_1},\mathscr{G}_{\tau_2}\}=0. \tag{2.9}$$

Moreover, Woodhouse demonstrated that for proper Riemannian and Lorentzian spaces the (apparently) more general concept of non-null separation ($g^{11}\neq 0$) is equivalent to either orthogonal separation or the existence of an ignorable variable. In this paper we shall use the Woodhouse construction (2.7) to give a complete characterisation of orthogonal separation.

Let $u(\boldsymbol{x},\boldsymbol{c})$ be a complete integral of the form (2.6) and denote by κ_1, κ_2, κ_3, respectively, the dimensions of the vector spaces spanned by elements (1): $\{\mathscr{G}_\tau, \tau\in I\}$, (2): $\{\mathrm{d}\mathscr{G}_\tau, \tau\in I\}$, (3): $\{\mathrm{d}_p\mathscr{G}_\tau, \tau\in I\}$ over (1) the field of real scalars and (2), (3), the field of functions $f(\boldsymbol{x})$, where I is an open interval on the real line on which $\mathscr{G}_\tau$ is defined. Here,

$$\mathrm{d}h(\boldsymbol{x},\boldsymbol{p})=\sum_i(\partial_{x^i}h\,\mathrm{d}x^i+\partial_{p_i}h\,\mathrm{d}p_i),\qquad \mathrm{d}_ph(\boldsymbol{x},\boldsymbol{p})=\sum_i\partial_{p_i}h\,\mathrm{d}p_i. \tag{2.10}$$

Lemma 3. $1\leq\kappa_3=\kappa_2\leq\kappa_1<\infty$.

The lemma is evident, except for the fact that $\kappa_3=\kappa_2$. It is evident that $\kappa_2\geq\kappa_3$. Let $\{\mathscr{G}_{\tau_l}, l=1,\ldots,\kappa_3\}$ be a set of κ_3 constants of the motion such that $\{\mathrm{d}_p\mathscr{G}_{\tau_l}\}$ is linearly independent over the field of functions $f(\boldsymbol{x})$. Then, by choosing new parameters

$\tilde{c}_j = z_j(\boldsymbol{c})$ with $\det(\partial\tilde{c}_j/\partial c_i) \neq 0$ if necessary, we can assume that the n constants of the motion λ_i associated with the parameters c_i in $u(\boldsymbol{x}, \boldsymbol{c})$ have the property $\lambda_l = \mathscr{G}_{\tau_l}$, $l = 1, \ldots, \kappa_3$. For fixed τ, $\mathrm{d}_p\mathscr{G}_\tau$ is a linear combination of the forms $\{\mathrm{d}_p\mathscr{G}_{\tau_l}, 1 \leqslant l \leqslant \kappa_3\}$ and $\mathrm{d}\mathscr{G}_\tau$ is a linear combination of the forms $\{\mathrm{d}\lambda_i, 1 \leqslant i \leqslant n\}$. Thus, there exist functions f_τ, g_τ such that

$$\mathscr{G}_\tau = f_\tau(\lambda_1, \ldots, \lambda_{\kappa_3}; \boldsymbol{x}) = g_\tau(\lambda_1, \ldots, \lambda_n). \tag{2.11}$$

Since $\mathrm{d}_pf_\tau = \mathrm{d}_pg_\tau$ and $\{\mathrm{d}_p\lambda_i, 1 \leqslant i \leqslant n\}$ is linearly independent we have

$$\partial g_\tau/\partial\lambda_i = 0, \qquad i = \kappa_3 + 1, \ldots, n, \tag{2.12}$$

so $\mathrm{d}\mathscr{G}_\tau$ is a linear combination of the forms $\{\mathrm{d}\mathscr{G}_{\tau_l}, 1 \leqslant l \leqslant \kappa_3\}$. Hence $\kappa_2 \leqslant \kappa_3$.

It follows from lemma 3 that

$$\mathscr{G}_\tau = \sum_{j=1}^{\kappa_1} f_j(\tau)\mathscr{G}_{\tau_j}, \qquad \tau \in I, \tag{2.13}$$

where $\{\mathscr{G}_{\tau_j}, j = 1, \ldots, \kappa_1\}$ is linearly independent (over the real scalars) and $\{\mathrm{d}_p\mathscr{G}_{\tau_i}, i = 1, \ldots, \kappa_3\}$ is linearly independent (over the field of functions on V^n). Thus

$$\mathscr{G}_{\tau_j} = g_j(\mathscr{G}_{\tau_1}, \ldots, \mathscr{G}_{\tau_{\kappa_3}}), \qquad j = \kappa_3 + 1, \ldots, \kappa_1. \tag{2.14}$$

Finally, we can add $\mathscr{H}$ to our basis set $\{\mathscr{G}_{\tau_j}\}$, eliminating one of the $\mathscr{G}_{\tau_i}$ if $\{\mathrm{d}_p\mathscr{H}, \mathrm{d}_p\mathscr{G}_{\tau_i}, i = 1, \ldots, \kappa_3\}$ is linearly dependent, so that (renumbering the τ_j if necessary)

$$\mathscr{G}_\tau = f_1(\tau)\mathscr{H} + \sum_{j=2}^{\kappa_1'} f_j(\tau)\mathscr{G}_{\tau_j}, \qquad \kappa_1' = \kappa_1 \text{ or } \kappa_1 + 1. \tag{2.15}$$

(It may be that $f_1(\tau) \equiv 0$.) Setting $\mathscr{H} = \mathscr{G}_{\tau_1}$ for convenience, we note that

$$\mathscr{G}_{\tau_l} = \mathscr{L}_l + \rho_l\mathscr{H}, \qquad l = 1, \ldots, \kappa_1', \tag{2.16}$$

where

$$\begin{aligned} &\rho_1 = 1, \qquad \rho_j = -[g^{11}(\tau_j, x^\gamma)]^{-1}, \qquad 2 < j < \kappa_1', \\ &\mathscr{L}_1 = 0, \qquad \mathscr{L}_j = \sum b_j^{\alpha\beta}(x^\gamma)p_\alpha p_\beta. \end{aligned} \tag{2.17}$$

Moreover, ρ_l is a root of $\mathscr{G}_{\tau_l}$ with respect to the Hamiltonian $\mathscr{H}$, and $\mathrm{d}x^1$ is the simultaneous eigenform for the $\{\mathscr{G}_{\tau_l}\}$ (Eisenhart 1949).

From (2.15) and the fact that $\mathscr{G}_{\tau_l}$ is a constant when evaluated at $p_j = \partial_{x^j}u(\boldsymbol{x}, \boldsymbol{c})$ we see that the *separation equations* for (2.1) in the coordinates (x^1, x^γ) are

$$p_1^2 + \sum_{j=1}^{\kappa_1'} f_j(x^1)\tilde{c}_j = 0, \qquad \mathscr{L}_l(x^\gamma, p_\alpha) + \rho_l(x^\gamma)c_1 - \tilde{c}_l = 0, \qquad 2 \leqslant l \leqslant \kappa_1', \tag{2.18}$$

where $\tilde{c}_j = c_j$, $1 \leqslant j \leqslant \kappa_3'$; $\tilde{c}_j = g_j(c_1, \ldots, c_{\kappa_3'})$, $\kappa_3' + 1 \leqslant j \leqslant \kappa_1'$.

We note that some of the separation parameters $\tilde{c}_j$ may be functions of the first κ_3' parameters. Indeed, if

$$\mathscr{H} = p_1^2 + f_2(x^1)p_2^2 + f_3(x^1)p_3^2 + f_4(x^1)p_2p_3$$

then (2.1) admits a complete integral

$$u(\boldsymbol{x}, \boldsymbol{c}) = u^{(1)}(x^1, \boldsymbol{c}) + \sqrt{c_2}x^2 + \sqrt{c_3}x^3$$

and separation equations

$$\begin{aligned} &p_1^2+c_1+f_2c_2+f_3c_3+f_4\sqrt{c_2c_3}=0,\\ &p_2^2 \quad -c_2 = 0,\\ &p_3^2 \quad -c_3 = 0,\\ &p_2p_3 \quad -\sqrt{c_2c_3} = 0, \end{aligned}$$

$\kappa_3' = \kappa_3+1=3$, $\kappa_1'=\kappa_1+1=4$.

It should be remarked that the functions $f_j(x^1)$, $\mathscr{L}_l(x^\gamma, p_\gamma)$, $\rho_l(x^\gamma)$ occurring in the canonical equations (2.18) are not arbitrary, for they are subject to the conditions $\{\mathscr{H}, \mathscr{G}_\tau\} = \{\mathscr{G}_\mu, \mathscr{G}_\tau\} = 0$ which are equivalent to

$$\{\mathscr{L}_l, \mathscr{L}_m\}=0, \qquad \{\mathscr{L}_l, \rho_m\} = \{\mathscr{L}_m, \rho_l\}, \qquad 2\leqslant l, m \leqslant \kappa_1'. \tag{2.19}$$

Now suppose (x^i) is a coordinate system on V^n with respect to which the Hamiltonian takes the form (2.8). We will determine necessary and sufficient conditions that the coordinate x^1 be partially separable, i.e. that the Hamilton–Jacobi equation admit a complete integral of the form (2.6). To do this we first form the function $\mathscr{G}_\tau$, (2.7), from the contravariant metric tensor. Clearly, necessary conditions for separation are that $\{\mathscr{H}, \mathscr{G}_\tau\}=0$, $\tau\in I$ and that $\kappa_2=\kappa_3$ where $\kappa_2 = \dim \mathrm{Span}\{\mathrm{d}\mathscr{G}_\tau, \tau\in I\}$, $\kappa_3 = \dim\mathrm{Span}\{\mathrm{d}_p\mathscr{G}_\tau, \tau\in I\}$ and the span is taken over the field of functions $f(\boldsymbol{x})$. These conditions are also sufficient. We see from (2.7) and (2.8) that

$$\begin{aligned} &\mathscr{H}(x^1, x^\gamma) = p_1^2/G(x^1, x^\gamma) + \mathscr{K}(x^1, x^\gamma, p_\alpha),\\ &\mathscr{G}_\tau = G(\tau, x^\gamma)\mathscr{K}(\tau, x^\gamma, p_\alpha) - \frac{G(\tau, x^\gamma)}{G(x^1, x^\gamma)}p_1^2 - G(\tau, x^\gamma)\mathscr{K}(x^1, x^\gamma, p_\alpha). \end{aligned} \tag{2.20}$$

Using these expressions and the condition $\{\mathscr{H}, \mathscr{G}_\tau\}=0$ it is straightforward to verify that $\{\mathscr{G}_\mu, \mathscr{G}_\tau\}=0$, $\mu, \tau\in I$. If $\kappa_2=\kappa_3$ then we can write

$$p_1^2 + f_1(x^1)\mathscr{H} + \sum_{j=2}^{\kappa_1'} f_j(x^1)\mathscr{G}_{\tau_j} = 0 \tag{2.21}$$

where $\{\mathrm{d}_p\mathscr{H}, \mathrm{d}_p\mathscr{G}_{\tau_2}, \ldots, \mathrm{d}_p\mathscr{G}_{\tau_{\kappa_3'}}\}$ is linearly independent and $\mathscr{G}_{\tau_i} = g_i(\mathscr{H}, \mathscr{G}_{\tau_2}, \ldots, \mathscr{G}_{\tau_{\kappa_3'}})$, $i=\kappa_3'+1, \ldots, \kappa_1'$. Then from lemma 2, there exists a complete integral $u(\boldsymbol{x}, \boldsymbol{c})$ of the Hamilton–Jacobi equation such that $\mathscr{H}=c_1$, $\mathscr{G}_{\tau_j}=c_j$, $2\leqslant j\leqslant\kappa_3'$, for $p_i = \partial_{x^i}u(\boldsymbol{x}, \boldsymbol{c})$. It follows immediately from (2.21) that $\partial_{x^\gamma}p_1 = \partial_{x^1}p_\gamma = 0$. Hence, u takes the form (2.6). We have proved

Theorem 1. Let $(x^1, \ldots, x^n)$ be a coordinate system such that

$$\mathscr{H}(\boldsymbol{x}, \boldsymbol{p}) \equiv g^{11}(\boldsymbol{x})p_1^2 + \sum_{\alpha,\beta=2}^{n} g^{\alpha\beta}(\boldsymbol{x})p_\alpha p_\beta.$$

The Hamilton–Jacobi equation is separable in the coordinate x^1 if and only if (1) $\{\mathscr{H}, \mathscr{G}_\tau\}=0$, $\tau\in I$ where

$$\mathscr{G}_\tau = \frac{-\mathscr{H}(x^1, x^\gamma, \boldsymbol{p}) + \Sigma\, g^{\alpha\beta}(\tau, x^\gamma)p_\alpha p_\beta}{g^{11}(\tau, x^\gamma)}$$

and (2) $\kappa_2=\kappa_3$.

We note that condition (2) is essential. Indeed, the example

$$\mathscr{H}=p_1^2+f_1(x^1)p_2^2+f_2(x^1)p_3^2+f_3(x^1)p_2p_3+f_4(x^1)(p_4p_3-p_5p_2)+f_5(x^1)(x^4p_2+x^5p_3)^2 \tag{2.22}$$

shows that (2) may be violated even when condition (1) holds.

3. The intrinsic characterisation

In this section we provide a coordinate-free characterisation of partial separation by one coordinate. (For simplicity of exposition we will express some of our formulae in terms of an arbitrary coordinate system $(\boldsymbol{y}, \boldsymbol{p})$ on $\tilde{V}^n$.)

Theorem 2. Let V^n be a pseudo-Riemannian space with Hamilton–Jacobi equation $\mathscr{H}=E$ and suppose:

(1) There exists a set $(\mathscr{G}_i, 2\leqslant i\leqslant \kappa)$ of second-order Killing tensors which are in involution $(\{\mathscr{H}, \mathscr{G}_i\}=\{\mathscr{G}_i, \mathscr{G}_j\}=0)$.

(2) The set $(\mathscr{H}, \mathscr{G}_1, \ldots, \mathscr{G}_\kappa)$ is linearly independent on $\tilde{V}^n$ (over the field of real scalars). dim Span$(\mathrm{d}\mathscr{H}, \mathrm{d}\mathscr{G}_1, \ldots, \mathrm{d}\mathscr{G}_\kappa)=$ dim Span$(\mathrm{d}_p\mathscr{H}, \mathrm{d}_p\mathscr{G}_1, \ldots, \mathrm{d}_p\mathscr{G}_\kappa)$, over the field of functions on V^n.

(3) The $(\mathscr{G}_i)$ admit a simultaneous eigenform $\omega=\Sigma_{j=1}^n \omega_j\,\mathrm{d}y^j$ (with respect to $\mathscr{H}=\Sigma g^{ij}p_ip_j$) corresponding to roots $\rho_1, \ldots, \rho_\kappa$. The eigenform ω is non-null: $\Sigma\, \omega_k g^{kj}\omega_j\neq 0$.

(4) There exist functions $h_1, \ldots, h_k$ on V^n, such that

$$\hat{\omega}^2=h_1\mathscr{H}+\sum_{i=2}^{k} h_i\mathscr{G}_i. \tag{3.1}$$

The h_i are unique. (Here, $\hat{\omega}^2$ is the function on $\tilde{V}^n$ whose local coordinate representation is $\hat{\omega}^2=\Sigma_{j,k=1}^n \omega^i\omega^j p_ip_j$.)

(5) For every pair (h_i, h_j) in (3.1) with $h_ih_j\neq 0$, $i\neq j$, then $\mathrm{d}(h_i/h_j)=Q(i,j)\omega$ where $Q(i,j)$ is a function on V^n.

Then there are two possibilities.

(a) There does not exist a non-zero pair (h_i, h_j), $i\neq j$ in (3.1) with $\mathrm{d}(h_i/h_j)$ non-zero. In this case there exists a coordinate system $(\boldsymbol{x}, \boldsymbol{p})$ on $\tilde{V}^n$ and a function g on V^n such that $\hat{\omega}^2=gp_1^2$ and $\{p_1, \mathscr{H}\}=0$. Hence x^1 is an ignorable coordinate: there is a complete integral of the form

$$u(\boldsymbol{x}, \boldsymbol{c})=c_2x^1+u^*(x^\alpha, \boldsymbol{c}). \tag{3.2}$$

(b) There exists at least one non-zero pair (h_i, h_j), $i\neq j$ in (3.1) with $\mathrm{d}(h_i/h_j)$ non-zero. In this case there is a function r on V^n such that $\omega=r\,\mathrm{d}x^1$ where x^1 is an orthogonal separable coordinate and each $\mathscr{G}_\tau$ associated with x^1 lies in the span of $\{\mathscr{H}, \mathscr{G}_i\}$ over the reals.

Conversely, if x^1 is an ignorable or an orthogonal separable coordinate, conditions (1)–(5) are satisfied by Killing tensors $\mathscr{G}_i$ whose relationship with x^1 is given by (a) or (b), respectively.

Proof. The converse follows easily from the results of § 2. Suppose on the other hand, conditions (1)–(5) hold and there is no non-zero pair (h_i, h_j) in (3.1) with $i\neq j$ and $\mathrm{d}(h_i/h_j)\neq 0$. Then either there is only one non-zero term h_i or all quotients h_i/h_j are constant. In either case, multiplying both sides of (3.1) by a non-zero function in V^n if necessary, we can assume that $\{\hat{\omega}^2, \mathscr{H}\}=0$. Thus there is a coordinate system $\boldsymbol{x}$ for V^n such that $\hat{\omega}^2=p_1^2$, hence a complete integral of the form (3.2).

Now suppose conditions (1)–(5) hold and there exists at least one pair (h_i, h_j) in (3.1) such that $\mathrm{d}(h_i/h_j)\neq 0$. It follows immediately from (5) that, after multiplication by a suitable non-zero function on V^n if necessary, we can assume that $\omega=\mathrm{d}x^1$ for a non-zero function x^1 on V^n. Moreover, there exists a coordinate system (x^1, x^α) with respect to which

$$\mathscr{H}=g^{11}p_1^2+\sum g^{\alpha\beta}p_\alpha p_\beta, \qquad \hat{\omega}^2=(g^{11})^2p_1^2, \tag{3.3}$$

and by (3)

$$\mathscr{G}_i=p_i\mathscr{H}+\mathscr{L}_i, \qquad \mathscr{L}_i=\Sigma\, \mathscr{G}_i^{\alpha\beta}p_\alpha p_\beta. \tag{3.4}$$

From (4) and (5) we see that

$$p_1^2=-\frac{1}{G}\left(f_1(x^1)\mathscr{H}+\sum_{i=2}^{k} f_i(x^1)\mathscr{G}_i\right), \tag{3.5}$$

where G is a non-zero function on V^n such that $\partial_{x^1}G=0$. Comparing coefficients of p_1^2 on both sides of (3.5) we find

$$\frac{-G(x)}{g^{11}}=\sum_{i=1}^{k} f_i(x^1)\rho_i(\boldsymbol{x})\neq 0, \qquad \rho_1=1. \tag{3.6}$$

Substituting expressions (3.3), (3.4) into conditions (1) we find $\partial_{x^1}\rho_i=\partial_{x^1}\mathscr{L}_i=0$, and

$$\{\mathscr{L}_l, \mathscr{L}_m\}=0, \qquad \{\mathscr{L}_l, \rho_m\}=\{\mathscr{L}_m, \rho_l\}, \qquad \{\mathscr{H}, \mathscr{L}_l\}=\{\rho_l, \mathscr{H}\}\mathscr{H}, \qquad 2\leqslant l, m\leqslant \kappa. \tag{3.7}$$

Now $\{p_1^2, \mathscr{L}_m\}=0$, and, substituting (3.5) into this expression and making use of (3.7), we find

$$\{\mathscr{L}_m, G\}\sum_{i=2}^{\kappa} f_i(x^1)\mathscr{L}_i=0, \qquad \{\mathscr{L}_m, G\}\sum_{i=1}^{\kappa}\rho_i(\boldsymbol{x})f_i(x^1)=0. \tag{3.8}$$

The second of equations (3.8) implies $\{\mathscr{L}_m, G\}=0$, $2\leqslant m\leqslant\kappa$; hence, G is a constant. We can assume $G\equiv 1$. From conditions (1), (2) and lemma 2 it follows that there exists a complete integral $u(\boldsymbol{x}, \boldsymbol{c})$ for the Hamilton–Jacobi equation such that each $\mathscr{G}_i$ is a function of the parameters $\boldsymbol{c}$ alone for $p_j=\partial_{x^j}u$. But then (3.5) for $G\equiv 1$ implies $\partial_{x^\gamma}p_1=\partial_{x^1}p_\gamma=0$, so x^1 is an orthogonal separable coordinate.

Condition (4) requires, in essence, that the $\mathscr{G}_i$ are related to the separable coordinate x^1 via the Woodhouse construction (2.15). The example $\mathscr{H}=p_1^2+p_2^2+p_3^2$, $\mathscr{G}_2=p_3(x^1p_2-x^2p_1)$, $\kappa=2$, satisfies conditions (1)–(3) but violates (4) because the common eigenform $x^1\,\mathrm{d}x^1+x^2\,\mathrm{d}x^2$ does not satisfy (3.1) for any h_1, h_2. Moreover, it is not possible to find an additional Killing tensor $\mathscr{G}_3$ such that $\{\mathscr{H}, \mathscr{G}_2, \mathscr{G}_3\}$ satisfies (3.1).

The example

$$\mathscr{H} = p_1^2 + h_1 p_2^2 + h_2 p_3^2 + h_3 p_2 p_3 + h_4(p_4 p_3 - p_5 p_2),$$
$$h_1 = f_1(x^1) + (x^4)^2 f_5(x^1), \qquad h_2 = f_2(x^1) + (x^5)^2 f_5(x^1), \tag{3.9}$$
$$h_3 = f_3(x^1) + 2x^4 x^5 f_5(x^1), \qquad h_4 = f_4(x^1),$$

the same as (2.22), satisfies conditions (1)–(4) but violates (5).

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